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Nonlinear Stability of Incoherence and Collective Synchronization in a Population of Coupled Oscillators

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A mean-field model of nonlinearly coupled oscillators with randomly distributed frequencies and subject to independent external white noises is analyzed in the thermodynamic limit. When the frequency distribution is *bimodal*, new results include subcritical spontaneous stationary synchronization of the oscillators, supercritical time-periodic synchronization, bistability, and hysteretic phenomena. Bifurcating synchronized states are asymptotically constructed near bifurcation values of the coupling strength, and their *nonlinear stability* properties ascertained.

KEY WORDS: Nonlinear oscillators; synchronization; mean-field model; bimodal distribution; bifurcation; nonlinear stability.

1. INTRODUCTION

A long-standing intellectual aspiration in many fields of science is to understand stable temporal and spatiotemporal phenomena in macroscopic systems from a microscopic point of view, perhaps by means of a statistical description. A relatively simple case in point is the self-synchronization of oscillator populations. Transition from incoherence to collective synchronization is a ubiquitous phenomenon in several fields of science: chemical processes,[(1)] biological organization,[(2,4)] and some models of dynamics of charge density waves in quasi-one-dimensional metals.[(3)]

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Winfree[2] first proposed to model the phenomenon of collective synchronization in terms of populations of coupled nonlinear oscillators, each having its own globally stable limit cycle. Kuramoto[5] then put forth a mathematically tractable model, containing all the main features of interest: Each oscillator runs at a frequency picked up from a given distribution, and all of them are coupled by a mean-field interaction. Thus, each member of the population tries to oscillate independently at its own frequency, while the coupling tends to synchronize it to all others. When the coupling is sufficiently weak, the oscillators run incoherently, while beyond some threshold, collective synchronization is established. In the limit of infinitely many oscillators, the degree of synchronization is measured by an order parameter that is nonzero in the synchronized state. This behavior is reminiscent of phase transitions in ferromagnetic materials (where the "disordering" role of the distribution of intrinsic frequencies is played by thermal white noise), whose dynamics was studied (with mean field coupling) by Desai and Zwanzig[6] and by Dawson.[7] New features introduced by the distribution of the intrinsic parameters of the oscillators can be appreciated in the dynamics of certain simple spin-glass models.[8] Strogatz and Mirollo[9] first studied rigorously the *linear* stability of the incoherent state in Kuramoto's model. They discovered that the incoherent state has pathological properties: The state with zero order-parameter is nonunique and is neutrally stable for coupling smaller than a certain critical value. Beyond such a critical value, the incoherent state becomes unstable and the synchronized state bifurcates from it. To elucidate these peculiarities, Strogatz and Mirollo added a small independent white-noise term to each oscillator. Such noise terms can be interpreted, e.g., as thermal fluctuations[3,6–8] or rapid fluctuations of the intrinsic frequencies of the oscillators.[9] In fact, synchronization of coupled oscillators having the same frequency but driven by white-noise sources has been studied in its own sake,[1,10,11] and even without assuming mean-field coupling.[12] See also ref. 13, where evidence is shown for synchronization of a distribution of oscillators subject to *hierarchical* (rather than mean-field) coupling.

In this paper we study the Kuramoto's model *with* noise, i.e.,

$$\dot{\theta}_i = \omega_i + \xi_i(t) + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \qquad i = 1, 2,..., N \tag{1.1}$$

$\xi_i(t)$ being independent white-noise processes with

$$\begin{aligned} \langle \xi_i(t) \rangle &= 0 \\ \langle \xi_i(s)\, \xi_j(t) \rangle &= 2D\, \delta_{ij}\, \delta(s-t) \end{aligned} \tag{1.2}$$

Here θ_i and ω_i represent the phase and the natural frequency of the ith oscillator, $K \geqslant 0$ is the coupling strength, N is the size of the population, and $D \geqslant 0$ represents the noise strength. Brackets denote an average over realizations of the noise. The frequencies ω_i are chosen at random from a distribution with density $g(\omega)$ which will be specified later.

The model (1.1)–(1.2) can be rewritten in a more convenient form by defining the order-parameter

$$re^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j} \tag{1.3}$$

Here $r(t) \geqslant 0$ measures the phase coherence of the oscillators, and $\psi(t)$ measures the average phase. In terms of (1.3), (1.1) becomes

$$\dot{\theta}_i = \omega_i + Kr \sin(\psi - \theta_i) + \xi_i(t), \qquad i = 1, 2,..., N \tag{1.4}$$

(1.4) is a system of coupled Langevin equations. In the limit of infinitely many oscillators, $N \to \infty$, it is possible to derive[14] a nonlinear Fokker–Planck equation for the one-oscillator probability density, $\rho(\theta, t, \omega)$,[9]

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial \theta^2} - \frac{\partial}{\partial \theta} (\rho v) \tag{1.5}$$

where the drift velocity is given by

$$v(\theta, t, \omega) = \omega + Kr \sin(\psi - \theta) \tag{1.6}$$

and the order-parameter amplitude $r(t)$ and phase $\psi(t)$ are given in terms of ρ and g by

$$re^{i\psi} = \int_0^{2\pi} \int_{-\infty}^{+\infty} e^{i\theta} \rho(\theta, t, \omega)\, g(\omega)\, d\theta\, d\omega \tag{1.7}$$

The probability density has to be 2π-periodic in the angle θ and normalized,

$$\int_0^{2\pi} \rho(\theta, t, \omega)\, d\theta = 1 \tag{1.8}$$

(1.5)–(1.8) can be formally derived by the following procedure sketched in ref. 14: Write $\rho(\theta, t, \omega)$ in terms of the N-oscillator probability density ρ_N solution of the linear Fokker–Planck equation associated to the system (1.4) for an initial condition where ρ_N is the product of N one-oscillator probability densities (molecular chaos). Then write the path integral representation of ρ_N in the resulting expression and perform approximately the integrals by means of the saddle point method in the limit $N \to \infty$. The resulting expression for $\rho(\theta, t, \omega)$ is then shown to obey (1.5)–(1.8). A rigorous study of the limit $N = \infty$ and proof of the previous formulas may follow from an extension of Dawson's procedures[7] to a system with varying frequencies such as (1.4).

Strogatz and Mirollo[9] considered (1.5)–(1.8) with a frequency distribution $g(\omega)$ which was even and nonincreasing, as did Kuramoto[5] before them. They showed that for $K < K_c$ the incoherent equiprobability distribution

$$\rho_0(\theta, t, \omega) \equiv \frac{1}{2\pi} \tag{1.9}$$

is *linearly* stable, and linearly unstable for $K > K_c$. At $K = K_c$ a new stationary solution (the partially synchronized state[5,9]) branches off (1.9). As $D \downarrow 0$, (1.9) is still unstable for $K > K_c$ [$= 2/\pi g(0)$ at $D = 0$], but it is neutrally stable for $K < K_c$: The whole spectrum of the equation linearized about (1.9) collapses to the imaginary axis. Furthermore, (1.9) is no longer the only solution of (1.5)–(1.8) with zero order-parameter: There are infinitely many "rotating waves" with $r = 0$.[9] Thus, the noise D could act as a small viscosity parameter selecting (1.9) as $D \downarrow 0$ among all the competing incoherent solutions with $r = 0$. (A different way of selecting one stable solution without adding noise is explained in the Appendix, which is based on work by Keller and Bonilla.[15])

In spite of the great advance in terms of understanding and methodology brought out by Strogatz and Mirollo's work, a few important problems remain. Paramount among them is the problem of *nonlinear stability* of both incoherent and synchronized solutions of (1.5)–(1.8) for $D > 0$ (and, more difficult, in the singular limit $D \downarrow 0$). Another question, not addressed previously, is the phenomenology that frequency distributions different from even nonincreasing functions may bring out. Indeed, some nontrivial new features appear when $g(\omega)$ has two maxima (bimodal distribution). In particular, changes in the character of bifurcating stationary solutions (from supercritical to subcritical) occur when the distance between the peaks of $g(\omega)$ surpasses a critical value. Again, in the dynamics of van Hemmen spin glasses, bimodal distributions change the stability and character of solutions bifurcating from the paramagnetic one [equivalent to our Eq. (1.9)].[8]

In this paper, we show that a bimodal distribution may drastically alter the class of stable solutions of the problem (1.5)–(1.8): Probability densities with a time-periodic order-parameter may appear. These features had been found previously only in more complicated models.[16] When the system is characterized by a bimodal distribution with a doubly delta-peaked frequency distribution, it can be viewed as representing a pair of coupled homogeneous populations, *each* exhibiting supercritical stationary synchronization when coupling is set to zero [except that, with this interpretation, the order-parameter (1.7) would not contain an integration over the frequency ω]. This point of view may help in understanding the appearance of time-periodic synchronization in terms of the desynchronization between the two subpopulations above, due to a too large difference in their collective frequencies.

Therefore, we address both types of problems: Nonlinear stability of solutions of (1.5)–(1.8) *and* new phenomenology brought out by bimodal distributions $g(\omega)$. Nonlinear stability is addressed by constructing bifurcating solutions and examining their stability with multiscale methods.[8,11] The rest of the paper is organized as follows. In Section 2, we consider the stationary states of the model, devoting special attention to the incoherent solution for which *linear* stability is studied. The *bimodal* frequency distribution leads to new phenomena that have not been observed previously: subcritical bifurcation from incoherence of stationary solutions and Hopf bifurcation as well. In Section 3, we construct the time-periodic branch of solutions bifurcating from incoherence and show that they are always *nonlinearly* stable. We stress that time-periodic solutions exist because the distribution is bimodal. The main results are finally summarized in Section 4. The Appendix contains a proof that the incoherent solution is stable in a weak sense among all solutions with zero order-parameter in the limits $D=0$, $N\to\infty$.

2. STATIONARY STATES AND LINEAR STABILITY

The *stationary* solutions of the nonlinear Fokker–Planck problem (1.5)–(1.8) can be represented by

$$\rho_0(\theta,\omega)=\frac{1}{Z(\omega)}\exp\left\{\frac{Kr}{D}\cos(\psi-\theta)\right\}$$
$$\times\int_0^{2\pi} d\theta_1 \exp\left\{-\frac{1}{D}\left[\omega\theta_1+Kr\cos(\psi-\theta-\theta_1)\right]\right\} \quad (2.1\text{a})$$

$$Z(\omega)=\int_0^{2\pi} d\theta_1 \exp\left\{\frac{Kr}{D}\cos(\psi-\theta_1)\right\}$$
$$\times\int_0^{2\pi} d\theta_2 \exp\left\{-\frac{1}{D}\left[\omega\theta_2+Kr\cos(\psi-\theta_1-\theta_2)\right]\right\} \quad (2.1\text{b})$$

Here, r and ψ are obtained by writing $\rho_0(\theta, \omega)$ instead of $\rho(\theta, t, \omega)$ in (1.7).[17]

Among these solutions, *the only one* characterized by a zero order-parameter ($r = 0$) is the incoherent solution,

$$\rho_0(\theta, \omega) \equiv \frac{1}{2\pi} \tag{2.2}$$

A natural question about the incoherent solution concerns its stability properties. The linear stability analysis has been done rigorously by Strogatz and Mirollo.[9] They found that only the discrete spectrum of the linearized problem is relevant to the stability issue. The eigenvalues λ are given by[9]

$$\frac{K}{2} \int_{-\infty}^{+\infty} \frac{g(v)}{\lambda + D + iv}\, dv = 1 \tag{2.3}$$

Further investigations based on (2.3) concerned only symmetric, one-humped, nonincreasing distributions $g(\omega)$. In this case there is *at most one* eigenvalue, which is necessarily *real*[9,18]; if it exists, then $\lambda > -D$. Moreover, for $K > K_c$ [a *critical* coupling, corresponding to $\lambda = 0$ in (2.3)], a new stationary state branches off from the *incoherent* solution, characterized by $r > 0$, corresponding to a *synchronized* state. The new solution is always stable because of the principle of exchange of stabilities in bifurcation theory.[19]

We now show that qualitatively new features appear when $g(\omega)$ is a *bimodal* distribution. In this case, (2.3) has *more than one* solution, λ, possibly complex. For the sake of concreteness, we shall confine ourselves to the following case[8]:

$$g(\omega) = \frac{1}{2}\left[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\right] \tag{2.4}$$

In this case, there are *two* eigenvalues explicitly given by

$$\lambda^{\pm} = -D + \frac{K}{4} \pm \frac{1}{4}(K^2 - 16\omega_0^2)^{1/2} \tag{2.5}$$

The stability boundaries for the incoherent solution can be calculated by equating to zero the greatest of Re λ^+ and Re λ^-. The result is depicted in Fig. 1.

When the coupling is small enough ($K < 2D$), the incoherent solution is linearly stable for all ω_0, whereas for a coupling strong enough ($K > 4D$), the incoherent solution is always linearly unstable. For intermediate

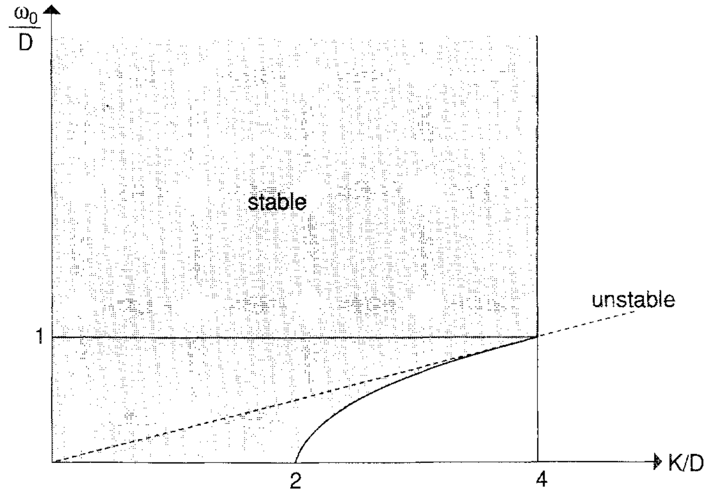

Fig. 1. Stability boundaries for the incoherent solution in the parameter space $(K/D, \omega_0/D)$ for the bimodal distribution (2.4). The dashed line separates the region where eigenvalues are real (below the line) from that where they are complex conjugate (above the line). Incoherence is linearly stable in the shadowed region and linearly unstable in the rest of the first quadrant.

couplings, $2D < K < 4D$, the incoherent solution may become linearly unstable in two different ways:

For $\omega_0 > D$, two complex conjugate eigenvalues cross the imaginary axis onto the right half-plane as K becomes larger than

$$K_c = 4D \tag{2.6}$$

At $K_c = 4D$ a branch of solutions with time-periodic order-parameter bifurcates from the incoherent solution. We shall construct this bifurcating branch and analyze its stability in the next section.

For $\omega_0 < D$ one eigenvalue in (2.5) becomes positive as K/D becomes larger than K_c/D, given by

$$\frac{K_c}{D} = 2\left(1 + \frac{\omega_0^2}{D^2}\right) \tag{2.7}$$

The stationary state that branches off the incoherent solution at K_c given by (2.7) can also be constructed by the method of the next section, but we shall use instead the more direct approach based on Eqs. (2.1) and (1.7) and the principle of exchange of stabilities for the same purpose [see ref. 17 for a similar construction with unimodal $g(\omega)$]. We insert (2.1) into the right-hand side of (1.7) (multiplied by $e^{-i\psi}$), and expand the resulting expression in powers of Kr/D. The result is

$$r = \frac{Kr}{2D}\left[\int_{-\infty}^{+\infty} d\omega \frac{g(\omega)}{1+\omega^2/D^2} - \frac{K^2r^2}{2D^2}\int_{-\infty}^{+\infty} d\omega \frac{1-2\omega^2/D^2}{(1+\omega^2/D^2)^2\,(4+\omega^2/D^2)}\, g(\omega) + O(r^4)\right] \quad (2.8)$$

According to the implicit function theorem, $r=0$ is an isolated solution of (2.8) for $K \neq K_c$, where

$$K_c = \frac{2D}{\int_{-\infty}^{+\infty} d\omega\, [g(\omega)/(1+\omega^2/D^2)]} \quad (2.9)$$

Note that (2.9) reduces to (2.7) when $g(\omega)$ is the distribution (2.4).

The coefficient of r^2 in (2.8) may be positive or negative, according to the shape of the frequency distribution $g(\omega)$. In fact, the integrand in this coefficient is $\eta(\omega/D)\, g(\omega)$, where

$$\eta(s) = \frac{1-2s^2}{(1+s^2)^2\,(4+s^2)} \quad (2.10)$$

From the shape of $\eta(s)$, depicted in Fig. 2, it is clear that the coefficient of r^2 on the right-hand side of (2.8) will be negative for unimodal distribu-

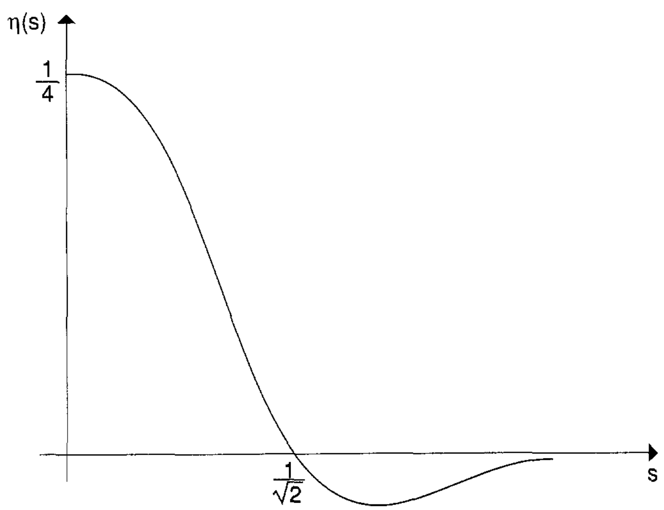

Fig. 2. The function $\eta(s) = (1-2s^2)/(1+s^2)^2\,(4+s^2)$ in the coefficient of r^2 that decides the direction of the bifurcating branch of stationary states.

tions $g(\omega)$, whereas it *may* become positive for a bimodal $g(\omega)$ with sufficiently separated peaks. Thus, for the distribution (2.4), (2.8) can be solved for $r^2>0$ and $K>K_c$ if $\omega_0<\omega_0^*$,

$$\omega_0^* = D/\sqrt{2} \tag{2.11}$$

The solution of (2.8) is then

$$r = \left(\frac{K-K_c}{K_c}\frac{D}{K_c}\frac{3+K_c/2D}{3-K_c/D}\right)^{1/2} + O(|K-K_c|), \qquad \omega < \frac{D}{\sqrt{2}} \tag{2.12}$$

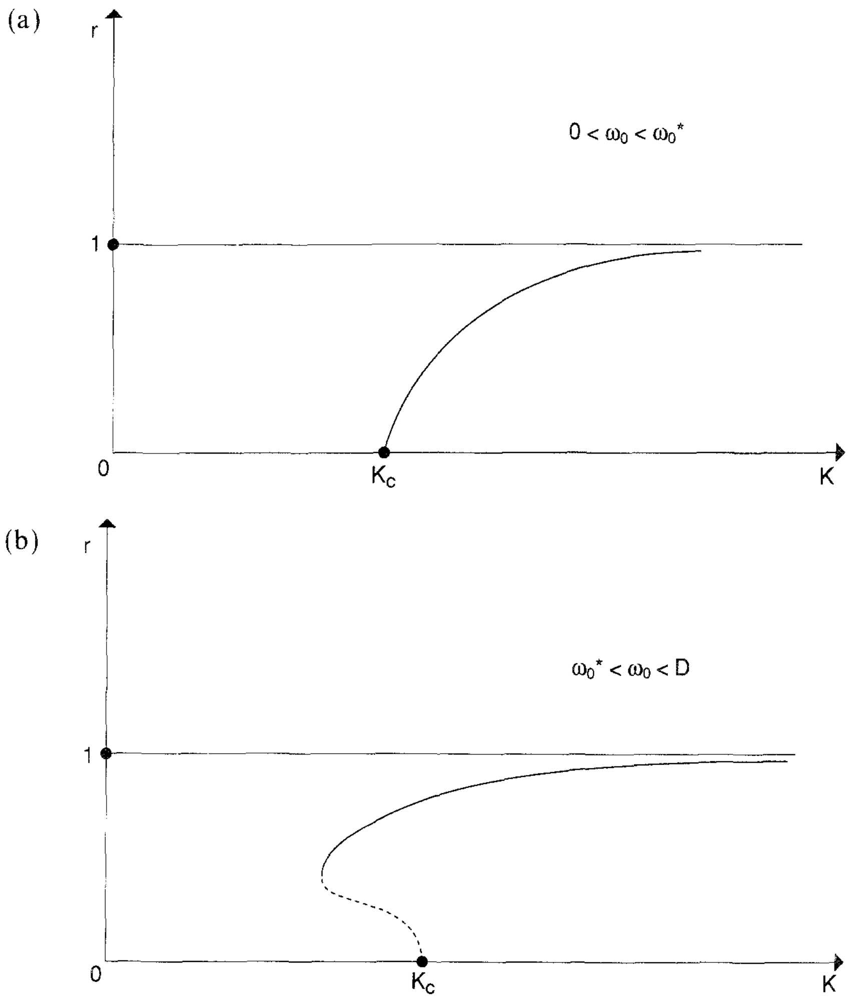

Fig. 3. (a) Supercritical stationary synchronization of the oscillators when $0<\omega_0<\omega_0^* = D/\sqrt{2}$. (b) Subcritical stationary synchronization for $D/\sqrt{2}<\omega_0<D$.

(2.12) is a supercritical bifurcating solution, and therefore it is stable;[19] see Fig. 3a. On the other hand, when $\omega_0 > \omega_0^*$, the solution of (2.8) exists only if $K < K_c$ and is therefore unstable[19] (Fig. 3b). The upper branch of stable stationary solutions in Fig. 3b can be found approximately for ω_0 close to ω_0^* by calculating explicitly the $O(r^4)$ term in (2.8).[11] The result of this straightforward calculation is that the coefficient of r^4 in (2.8), evaluated at $\omega = \omega_0^*$, is negative, which confirms the picture in Fig. 3b. That $r \uparrow 1$ as $K \to +\infty$ for any $\omega_0 > 0$ is straightforwardly proven by evaluating the integrals in (1.7) and in the stationary solution (2.1) by means of Laplace's method.

3. TIME-PERIODIC STATES

In the previous section we have shown that the incoherent solution becomes linearly unstable with purely imaginary eigenvalues at $K_c = 4D$ when $\omega_0 > D$. Here we shall construct the branch of time-periodic solutions of (1.5)–(1.8) that bifurcates from the incoherent solution at $K = K_c$. We use the two-time-scale method as in Kogelman and Keller,[20] which not only yields an approximate expression for the bifurcating solution, but also gives its *nonlinear stability* properties.

Let us define the small parameter ε that measures the departure from the critical value $K_c = 4D$ by

$$K = K_c + \varepsilon^2 K_2, \qquad 0 < \varepsilon \ll 1 \tag{3.1}$$

$K_2 = \pm 1$ has to be determined later according to the direction of the bifurcating branch. The definition (3.1) will be justified later. The probability density $\rho(\theta, t, \omega; \varepsilon)$ will be sought for according to the Ansatz[8,11]

$$\rho(\theta, t, \omega; \varepsilon) = \frac{1}{2\pi} \exp\left\{ \sum_{j=1}^{3} \varepsilon^j \sigma_j(\theta, t, \tau, \omega) + O(\varepsilon^4) \right\} \tag{3.2}$$

$$\tau = (K - K_c)t = \varepsilon^2 K_2 t \tag{3.3}$$

Near $K = K_c$, small disturbances from the incoherent solution decay or grow according to the values of the factor

$$\exp\{\lambda(K)t\} \sim \exp\left\{ \operatorname{Re} \frac{\partial \lambda(K_c)}{\partial K} (K - K_c)t + i \operatorname{Im} \lambda(K_c)t \right\} \tag{3.4}$$

where $\lambda(K)$ is the complex eigenvalue of Section 2 for which $\operatorname{Re} \lambda(K_c) = 0$. Equation (3.4) motivates the introduction of the slow and fast time scales (3.3) and t, respectively. The exponential Ansatz (3.2) was introduced in ref. 14 (see also refs. 8 and 11) motivated by the failure of the usual expansion of ρ in power series of ε: An algebraic Ansatz yields a vertical bifurcating branch to all orders in ε.[11]

We now insert (3.1)–(3.3) in the governing equations (1.5)–(1.8) and equate terms of equal order in ε in the resulting expressions. Thus, we obtain a hierarchy of equations which, up to $O(\varepsilon^4)$, is

$$L\sigma_1 \equiv (\partial_t - D\partial_\theta^2 + \omega\partial_\theta)\,\sigma_1 - K_c \operatorname{Re} e^{-i\theta}\langle e^{i\theta'}, \sigma_1\rangle = 0 \qquad (3.5\text{a})$$

$$\int_0^{2\pi} \sigma_1(\theta, t, \tau, \omega)\, d\theta = 0 \qquad (3.5\text{b})$$

$$L\sigma_2 = -L\frac{\sigma_1^2}{2} - K_c\partial_\theta\{\sigma_1 \operatorname{Im} e^{-i\theta}\langle e^{i\theta'}, \sigma_1\rangle\} \qquad (3.6\text{a})$$

$$\int_0^{2\pi} \sigma_2(\theta, t, \tau, \omega)\, d\theta = -\frac{1}{2}\int_0^{2\pi} [\sigma_1(\theta, t, \tau, \omega)]^2\, d\theta \qquad (3.6\text{b})$$

$$L\sigma_3 = -K_2[\partial_\tau\sigma_1 - \operatorname{Re} e^{-i\theta}\langle e^{i\theta'}, \sigma_1\rangle] - L\left(\sigma_1\sigma_2 + \frac{1}{6}\sigma^3\right) \qquad (3.7\text{a})$$

$$-K_c\,\partial_\theta\left\{\sigma_1 \operatorname{Im} e^{-i\theta}\left\langle e^{i\theta'}, \sigma_2 + \frac{\sigma_1^2}{2}\right\rangle + \left(\sigma_2 + \frac{\sigma_1^2}{2}\right)\operatorname{Im} e^{-i\theta}\langle e^{i\theta'}, \sigma_1\rangle\right\}$$

$$\int_0^{2\pi} \sigma_3(\theta, t, \tau, \omega)\, d\theta = -\int_0^{2\pi}\left(\sigma_1\sigma_2 + \frac{1}{6}\sigma_1^3\right) d\theta \qquad (3.7\text{b})$$

In Eqs. (3.5)–(3.7) we have defined the scalar product:

$$\langle \alpha(\theta, \omega), \beta(\theta, \omega)\rangle = \frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^{+\infty} \alpha(\theta, \omega)\,\beta(\theta, \omega)\, g(\omega)\, d\omega\, d\theta \qquad (3.8)$$

The solution of the homogeneous linear equation (3.5) is given by[9]

$$\sigma_1(\theta, t, \tau, \omega) = \frac{A(\tau)}{D + i(\omega + \Omega)}\, e^{i(\Omega t + \theta)} + \text{cc} \qquad (3.9)$$

plus terms that decay exponentially on the fast time scale. These terms correspond to the rest of the spectrum (discrete *and* continuous) of the operator L.[9] Since we are interested in studying stability, therefore in the long-time limit $t \to +\infty$, we neglect these terms. In (3.9), "cc" means the complex conjugate of the preceding term, and $A(\tau)$ is an as yet undetermined complex function of the slow time τ. At $K = K_c$, the eigenvalues $\lambda^\pm$ become

$$\lambda^\pm = \pm i\Omega, \qquad \Omega \equiv (\omega_0^2 - D^2)^{1/2} \qquad (3.10)$$

Inserting (3.9) in the right-hand side of (3.6a), we see that it is proportional to $e^{i2(\Omega t+\theta)}$. Thus, we seek a solution σ_2 of the form

$$\sigma_2 = B_1(\tau) + B_2(\tau)\, e^{i2(\Omega t+\theta)} + \mathrm{cc} \tag{3.11}$$

B_1 and B_2 are determined from Eq. (3.6), with the result

$$\sigma_2(\theta, t, \tau, \omega) = -\frac{|A(\tau)|^2}{D^2+(\Omega+\omega)^2} + \frac{1}{2}\left[3 - \frac{D}{D+i(\Omega+\omega)}\right] \times \frac{A^2(\tau)\, e^{i2(\Omega t+\theta)}}{[D+i(\Omega+\omega)][2D+i(\Omega+\omega)]} + \mathrm{cc} \tag{3.12}$$

We now try to solve (3.7) by introducing (3.9) and (3.12) into the right-hand side of (3.7), which contains terms proportional to $e^{i(\Omega t+\theta)}$, $e^{i3(\Omega t+\theta)}$, and their cc. The term proportional to $e^{i(\Omega t+\theta)}$ yields in general secular terms (which are unbounded on the fast time scale) in the corresponding part of σ_3, say $\sigma_3^{(1)}$. We therefore choose the term proportional to K_2 so as to eliminate these secular terms. Notice that this explains the scaling (3.1). To eliminate the secular terms, we need a solution of the equation

$$L\sigma_3^{(1)} = Q_1(\tau, \omega)\, e^{i(\Omega t+\theta)} \tag{3.13}$$

that is 2π-periodic in θ and bounded in t, that is,

$$\sigma_3^{(1)} = P_1(\tau, \omega)\, e^{i(\Omega t+\theta)} \tag{3.14}$$

$Q_1(\tau, \omega)$ is obtained by inserting (3.9) and (3.12) in (3.7), and is given by

$$Q_1(\tau, \omega) = -K_2\left(\frac{dA/d\tau}{D+i(\Omega+\omega)} - \frac{A}{2D}\right) + \left\{2D\langle 1, \chi\rangle - [D+i(\Omega+\omega)]\,\chi(\omega) - \frac{1}{[D+i(\Omega+\omega)][2D+i(\Omega+\omega)]}\right\} A\,|A|^2 \tag{3.15a}$$

$$\chi(\omega) = \frac{i(\Omega+\omega)}{[D^2+(\Omega+\omega)^2][D+i(\Omega+\omega)][2D+i(\Omega+\omega)]} \tag{3.15b}$$

We determine P_1 by substitution of (3.14) into (3.13),

$$[D+i(\Omega+\omega)]\, P_1 - \frac{K_c}{2}\langle 1, P_1\rangle = Q_1$$

Then we can solve for P_1:

$$P_1 = \frac{K_c\langle 1, P_1\rangle}{2[D+i(\Omega+\omega)]} + \frac{Q_1}{D+i(\Omega+\omega)} \tag{3.16}$$

From (2.3), we know that $\frac{1}{2}K_c\langle 1, 1/[D+i(\Omega+\omega)]\rangle = 1$, so that the scalar product of 1 with (3.16) gives

$$\left\langle \frac{1}{D+i(\Omega+\omega)}, Q_1(\omega, \tau) \right\rangle = 0 \tag{3.17}$$

(cf. ref. 9, p. 624). Equation (3.17) is a nonresonance condition: When it holds, no secular terms appear in σ_3. From (3.15) and (3.17), we obtain the following evolution equation for $A(\tau)$:

$$\frac{dA}{d\tau} = \left(\lambda_1 - \frac{\gamma}{K_2}|A|^2\right) A \tag{3.18a}$$

$$\lambda_1 \equiv \frac{(1/2D)\langle 1, 1/[D+i(\Omega+\omega)]\rangle}{\langle 1, 1/[D+i(\Omega+\omega)]^2\rangle} = \frac{\partial\lambda(K_c)}{\partial K} = \frac{1}{4}\left(1 - i\frac{K_c}{\Omega}\right) \tag{3.18b}$$

$$\gamma \equiv \frac{\langle 1, 1/[D+i(\Omega+\omega)]^2\,[2D+i(\Omega+\omega)]\rangle}{\langle 1, 1/[D+i(\Omega+\omega)]^2\rangle} = \frac{\Omega\{2\Omega(4\omega_0^2+9D^2) - iD(14\omega_0^2 - 9D^2)\}}{D(16\omega_0^2+9D^2)} \tag{3.18c}$$

(3.18) has the following periodic solution:

$$A(\tau) = Re^{i\psi_0(\tau-\tau_0)} \tag{3.19a}$$

$$R = \left(K_2 \frac{\operatorname{Re}\lambda_1}{\operatorname{Re}\gamma}\right)^{1/2} \tag{3.19b}$$

$$\psi_0 = \operatorname{Im}\lambda_1 - \operatorname{Im}\gamma\frac{\operatorname{Re}\lambda_1}{\operatorname{Re}\gamma} \tag{3.19c}$$

Note that γ is a function of D and ω_0. Since $\Omega^2 = \omega_0^2 - D^2 > 0$, $\operatorname{Re}\gamma > 0$ in (3.18c). Then $K_2 = 1$ and (3.19) exists for $K > K_c$. An explicit solution of (3.18a) shows that any nonzero initial value $A(0)$ tends to (3.19a) as $\tau = \varepsilon^2 t \to +\infty$. Thus, (3.19) corresponds to the following periodic solution of (1.5)–(1.8) (except for a phase shift τ_0) which is asymptotically stable:

$$\rho(\theta, t, \omega) \sim \frac{1}{2\pi}\exp\left\{|K-K_c|^{1/2}\frac{R}{D+i(\Omega+\omega)} \times \exp\{i[(\Omega+\varepsilon^2 K_2\psi_0)t+\theta]\} + \text{cc}\right\} \tag{3.20}$$

as $K \downarrow K_c$ ($\operatorname{Re}\gamma > 0$, $K_2 = 1$). This situation is depicted in Fig. 4.

Had $\operatorname{Re}\gamma$ been negative in (3.18c) [which is not the case for the bimodal $g(\omega)$], $K_2 = -1$, (3.19b), and (3.19a) would be reached from any initial condition $A(0) \neq 0$ as $\tau = -\varepsilon^2 t \to +\infty$. The limit of long *positive* times corresponds to $\tau \to -\infty$, and then $A(\tau) \to 0$, which means that the periodic solution (3.20) (now existing for $K < K_c$) is unstable if $\operatorname{Re}\gamma < 0$.

Notice that a small-amplitude equation of the form of (3.18a) was earlier derived for similar problems in ref. 21, where, however, there was no frequency distribution.

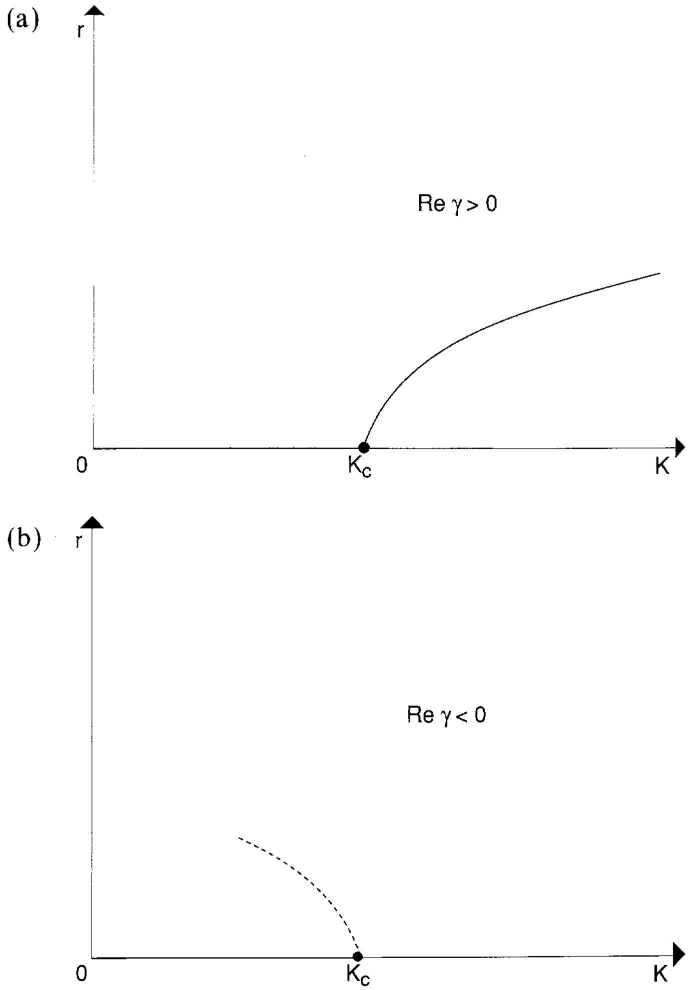

Fig. 4. (a) Supercritical Hopf synchronization of the oscillators when $\operatorname{Re}\gamma > 0$, $K_c = 4D$, $\omega_0 > D$. (b) Subcritical bifurcation for $\operatorname{Re}\gamma < 0$ [*not* realized by the special bimodal distribution (2.4)].

4. SUMMARY AND CONCLUSIONS

We have analyzed the synchronization of an infinite population of oscillators with randomly distributed frequencies, coupled nonlinearly and subject to external white noises. While this problem was studied earlier by several authors for unimodal frequency distribution functions $g(\omega)$,[5,9,17] we have found qualitatively new results for a bimodal $g(\omega)$. First of all, the *linear* stability analysis of the incoherent solution (where oscillators are desynchronized) indicates that, for oscillator couplings K large enough, incoherence may become unstable both steadily and oscillatorily in time (Fig. 1). For unimodal $g(\omega)$, only stationary solutions could bifurcate from incoherence.[9] For our bimodal $g(\omega)$, a stable stationary solution with $r > 0$ (representing synchronization of the oscillators) branches off incoherence for $K > K_c$ if the separation between the peaks of $g(\omega)$, $2\omega_0$, is small enough (Fig. 3a). When $\omega_0 \in (D/\sqrt{2}, D)$, the synchronized stationary state branches off *subcritically* from incoherence (Fig. 3b). Finally, when $\omega_0 > D$, a stable synchronized state with time-periodic order-parameter bifurcates from incoherence for $K > K_c = 4D$ (Fig. 4a).

We have constructed all bifurcating solutions in a neighborhood of the critical coupling K_c by using explicit formulas in the case of stationary solutions or by means of multitime scales in the case of oscillatory solutions. The latter method also yields the nonlinear stability or instability of the solutions. Putting together all our findings, we conjecture the bifurcation diagram of Fig. 5 for $\omega_0 > D$. Further study is needed to ascertain

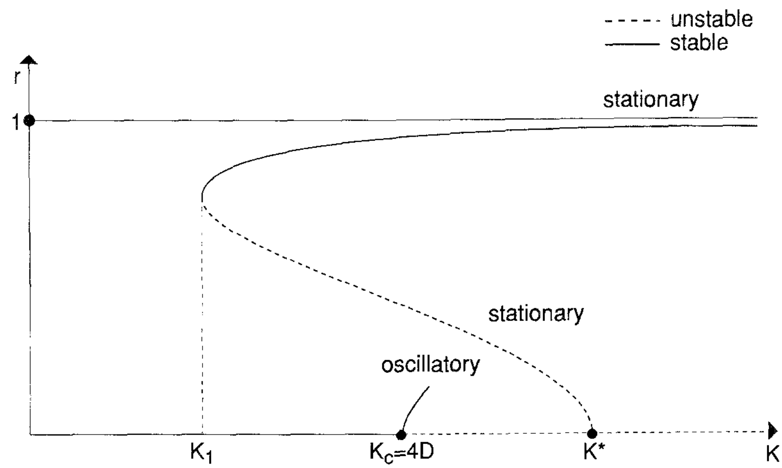

Fig. 5. Conjectured global bifurcation diagram for $\omega_0 > D$, indicating coexistence of stable stationary and time-periodic solutions for certain values of the coupling constant K.

where the Hopf branch of time-periodic solutions ends. Notice that coexistence of stable stationary solutions exists for $K_1 < K < K_c = 4D$, while for $K > K_c$ there is bistability between synchronized stationary and time-periodic states.

After this paper was completed, numerical Brownian simulations conducted by J. M. Casado showed evidence for the stability of the synchronized stationary solutions, at least for $K \gg 4D$. These simulations show an interesting transient behavior of the model for intermediate values of K that deserves further analysis.

APPENDIX. STABILITY (IN THE WEAK SENSE) OF INCOHERENCE WHEN $D = 0$

In this Appendix we show that, among all solutions with $r = 0$ of the model (1.1) with $D = 0$ and $N \to \infty$, incoherence is stable in a weak sense as $t \to \pm\infty$. If $r = 0$, $D = 0$, (1.1) describes a collection of oscillators rotating at constant frequencies, which is the model considered in ref. 15. Consider the fraction of oscillators having their angles between θ and $\theta + d\theta$ and frequencies between ω and $\omega + d\omega$:

$$\rho_N(\theta, t, \omega) = \frac{1}{N} \sum_{n=-N/2}^{N/2} \sum_{m=-\infty}^{\infty} \delta(\theta_n(0) + \omega_n t - \theta - 2\pi m)\, \delta(\omega_n - \omega) \qquad \text{(A.1)}$$

Notice that ρ_N is a 2π-periodic function of θ with this definition. We want to show that ρ_N tends to $1/2\pi$ in a weak sense, i.e., that as $N \to \infty$ and $t \to \infty$,

$$(\rho_N, \psi) \equiv \int_{-\infty}^{+\infty} d\omega\, \rho_N(\theta, t, \omega)\, \psi(\theta, \omega) \to \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega\, g(\omega)\, \psi(\theta, \omega) \qquad \text{(A.2)}$$

for all $\psi(\theta, \omega)$ smooth enough.

In fact, by using (A.1) in (A.2), we get

$$(\rho_N, \psi) = \frac{1}{N} \sum_{n=-N/2}^{N/2} \sum_{m=-\infty}^{\infty} \delta(\theta_n(0) + \omega_n t - \theta - 2\pi m)\, \psi(\theta, \omega_n)$$

In the continuum limit, $N \to \infty$, the sum over the oscillators becomes an integral over the frequencies with density $g(\omega)$:

$$\frac{1}{N} \sum_{n=-N/2}^{N/2} \to \int_{-\infty}^{+\infty} d\omega\, g(\omega)$$

so that

$$(\rho_N, \psi) \to \int_{-\infty}^{+\infty} \sum_{m=-\infty}^{\infty} \delta(\theta_n(0) + \omega t - \theta - 2\pi m)\, \psi(\theta, \omega)\, g(\omega)\, d\omega$$

$$= \frac{1}{t} \sum_{m=-\infty}^{\infty} \psi\left(\theta, \frac{2\pi m + \theta - \theta_n(0)}{t}\right) g\left(\frac{2\pi m + \theta - \theta_n(0)}{t}\right) \tag{A.3}$$

As $t \to \infty$, the sum in (A.3) converges to a Riemann integral. If we write $dy = 2\pi/t$, we obtain from (A.3) the result

$$\lim_{t \to \pm\infty} \lim_{N \to \infty} (\rho_N, \psi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dy\, g(y)\, \psi(\theta, y) \tag{A.4}$$

We have therefore proved that the weak limit as $t \to \pm\infty$ for the density functions ρ_N $(N \to \infty)$ that have $r = 0$ is the incoherent distribution $1/2\pi$.

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